The Spectral Geometry of Flat Disks

Vol. 61, No. 1 DUKE MATHEMATICAL JOURNAL © August 1990

THE SPECTRAL GEOMETRY OF FLAT DISKS

ROBERT BROOKS, YAKOV ELIASHBERG AND C. MCMULLEN

In [6], Mark Kac raised the question of whether two domains in the Euclidean plane $\mathbb{R}^2$ which have the same spectrum of the Laplacian are congruent. Central to his approach to this question was the fact that certain invariants of the Laplacian for such a domain D, called the heat invariants, are expressible in terms of the area of D and integrals over the boundary ∂D of the domain of terms involving the length, the geodesic curvature, and derivatives of these quantities—see §1 below for a discussion.

One is naturally led by this approach to consider the extent to which the geometry of D in a neighborhood of ∂D governs the spectrum of D. To that end, let us say that two flat disks D_1 and D_2 are isometric near the boundary if there are neighborhoods of ∂D_1 and ∂D_2 which are isometric. Furthermore, we will say that D_1 and D_2 are piecewise isometric near the boundary if there are neighborhoods N_1 and N_2 of ∂D_1 and ∂D_2 such that N_1 can be subdivided into finitely many pieces and rearranged to obtain N_2.

It is easy to see that if D_1 and D_2 are piecewise isometric near the boundary and have the same area, then all of the heat invariants of D_1 and D_2, and indeed all such integrals over the boundary, must agree. It is also easy to see that if D_1 and D_2 are planar disks which are isometric near the boundary, then D_1 and D_2 are congruent.

However, it is an interesting fact that there are flat disks D_1 and D_2 which are isometric near the boundary, but which are not themselves isometric. Such disks immerse into the plane so that they have a common boundary curve. We will see how to construct such examples in §2 below.

In §3 and §4 below, we will then show:

THEOREM 1. *There are compact flat disks D_1 and D_2 which are isometric near the boundary, but which are not isospectral for either Dirichlet or Neumann boundary conditions.*

Using similar techniques, we will also show:

THEOREM 2. *There exist planar disks D_1 and D_2 which are piecewise isometric near the boundary, but which are not isospectral for either Dirichlet or Neumann boundary conditions.*

Finally, in §5 we will establish the analogue of Theorem 1 in dimensions greater than 2.

Received August 12, 1989.

It is a pleasure to thank the Mathematical Sciences Research Institute for its support during the development of this work.

1. The Heat Invariants. Let D be a compact, connected Riemannian manifold, with boundary ∂D. A choice of self-adjoint boundary conditions (for our purposes, either Dirichlet or Neumann conditions) turns the Laplacian Δ into a positive semi-definite operator, with discrete spectrum tending to $+\infty$. We denote the spectrum by

$$0 = \lambda_0^N(D) < \lambda_1^N(D) \leqslant \lambda_2^N(D) \leqslant \cdots$$

for Neumann conditions, and

$$0 = \lambda_0^D(D) < \lambda_1^D(D) \leqslant \lambda_2^D(D) \leqslant \cdots$$

for Dirichlet conditions.

The Heat Kernel $H_t^N(x, y)$(resp. $H_t^D(x, y)$) is defined by

$$H_t^N(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^N(x)\phi_i^N(y)$$

$$H_t^D(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^D(x)\phi_i^D(y)$$

where ϕ_i^N(resp. ϕ_i^D) runs over an orthonormal basis of eigenfunctions with eigenvalue λ_1.

The trace of the Heat Kernel is defined by

$$tr(H_t^N) = \int_D H_t^N(x, x)\, dx$$

$$tr(H_t^D) = \int_D H_t^D(x, x)\, dx$$

and has a well-known asymptotic expansion as $t \to 0$ given by

$$tr(H_t^N) \sim \frac{1}{(4\pi t)^{n/2}}\left(\sum_{i=0}^{\infty} (a_{i/2}^N)t^{i/2}\right)$$

$$tr(H_t^D) \sim \frac{1}{(4\pi t)^{n/2}}\left(\sum_{i=0}^{\infty} (a_{i/2}^D)t^{i/2}\right)$$

where the a_i^N (resp. a_i^D) are given by integrating local expressions in the curvature over D and local expressions in the geodesic curvature over ∂D, and $n = \dim(D)$.

The case of Dirichlet boundary conditions has been studied extensively, but there is little difference of an analytic nature between the two cases–see [3] for a general reference. In what follows, if we do not specify Dirichlet or Neumann boundary conditions, it will be understood that Dirichlet conditions are intended, but that an analogous formula would be possible using Neumann conditions.

In the case when D is flat, all the curvature terms disappear, so that we have, when $\dim(D) = 2$,

$$a_0(D) = \text{area}(D)$$

$$a_{1/2}(D) = -\frac{1}{\sqrt{8\pi}}\text{length}(\partial D)$$

$$a_1(D) = \frac{\chi(D)}{6},$$

where $\chi(D)$ is the Euler characteristic of D (note that this last is an integral over ∂D, by the Gauss-Bonnet Theorem with boundary).

The a_i's are called the heat invariants of D, and are visibly spectral invariants of D. It is evident that if D_1 and D_2 have the same area and are isometric near the boundary, or even piecewise isometric near the boundary, then all of the a_i's agree for D_1 and D_2, since they are expressed by integrals which agree piecewise.

One might be tempted to believe that the a_i's determine a lot of the geometry of D. Indeed, this temptation seems to have motivated Kac in [6]. That this is partially true was demonstrated by Melrose in [8], who showed that when D is a plane domain, the a_i^D's determine ∂D (and hence D) up to a compact set of possibilities. However, this compact set may include certain degeneracies in ∂D, as in the following picture:

It is clear that one may pinch the middle of this region so as to keep the a_i's bounded. However, the region D tends in the limit to two disks joined at a point.

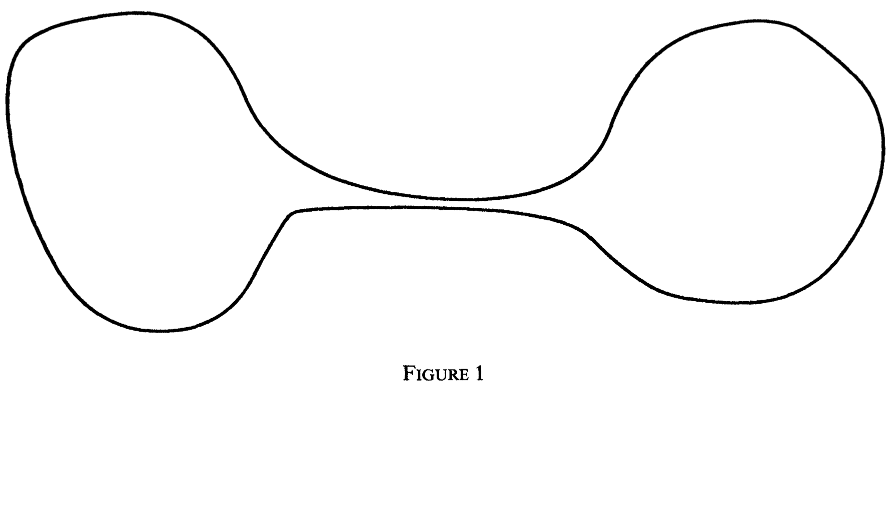


FIGURE 1

The question was then raised as to whether this pinching could be detected spectrally.

See [11] for a discussion of this question. In particular, it is shown there and in [9] how the determinant of the Laplacian serves as a non-local spectral invariant which detects this kind of degeneracy. We will return to this line of thought in §4 below.

2. Flat Disks. Let D be a compact flat two-dimensional manifold with boundary ∂D. Then there is an isometric immersion $h: \tilde{D} \to \mathbb{R}^2$ of the universal cover $\tilde{D}$ of D into the Euclidean plane, which is unique up to Euclidean motion, and is described as follows: given a point $x \in \tilde{D}$, there is an isometry between a neighborhood of x and a neighborhood of a point in $\mathbb{R}^2$. This isometry can then be extended uniquely along all paths in $\tilde{D}$ to give h.

When D is a disk, then $\tilde{D} = D$, and so h gives an immersion of D into $\mathbb{R}^2$. The restriction of h to ∂D is then a closed curve γ in the plane.

If γ has no self-intersection, then the Jordan Curve Theorem says that γ bounds a unique disk, which must then be D, and we have exhibited D as a plane domain. However, if γ has self-intersections, then the situation is more complicated, and in particular there could conceivably be more than one immersed disk D which has γ as a boundary.

Such a curve was studied by Blank [1], who attributed the example to Milnor (see [7] for a historical discussion), and independently by Eliashberg [5], and is shown below in Figure 2:

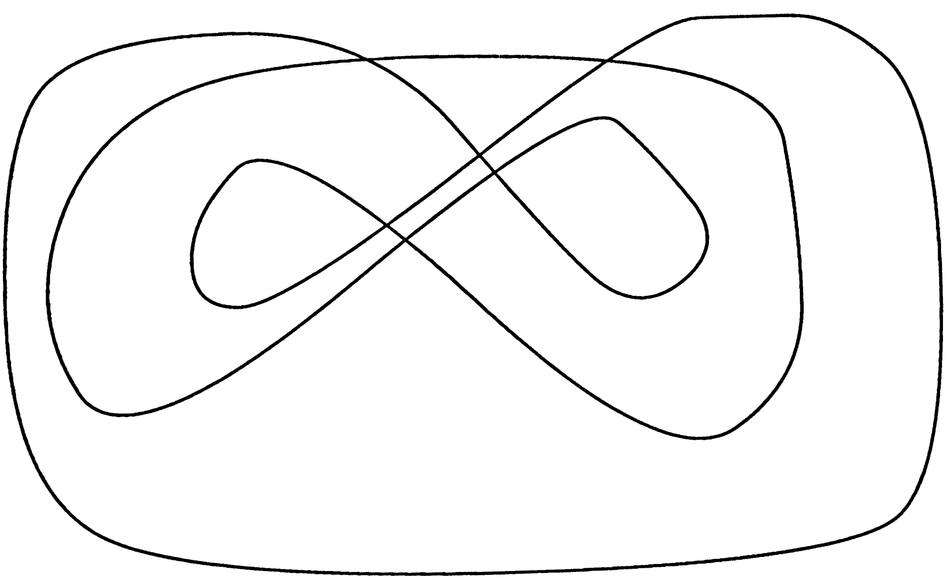


FIGURE 2

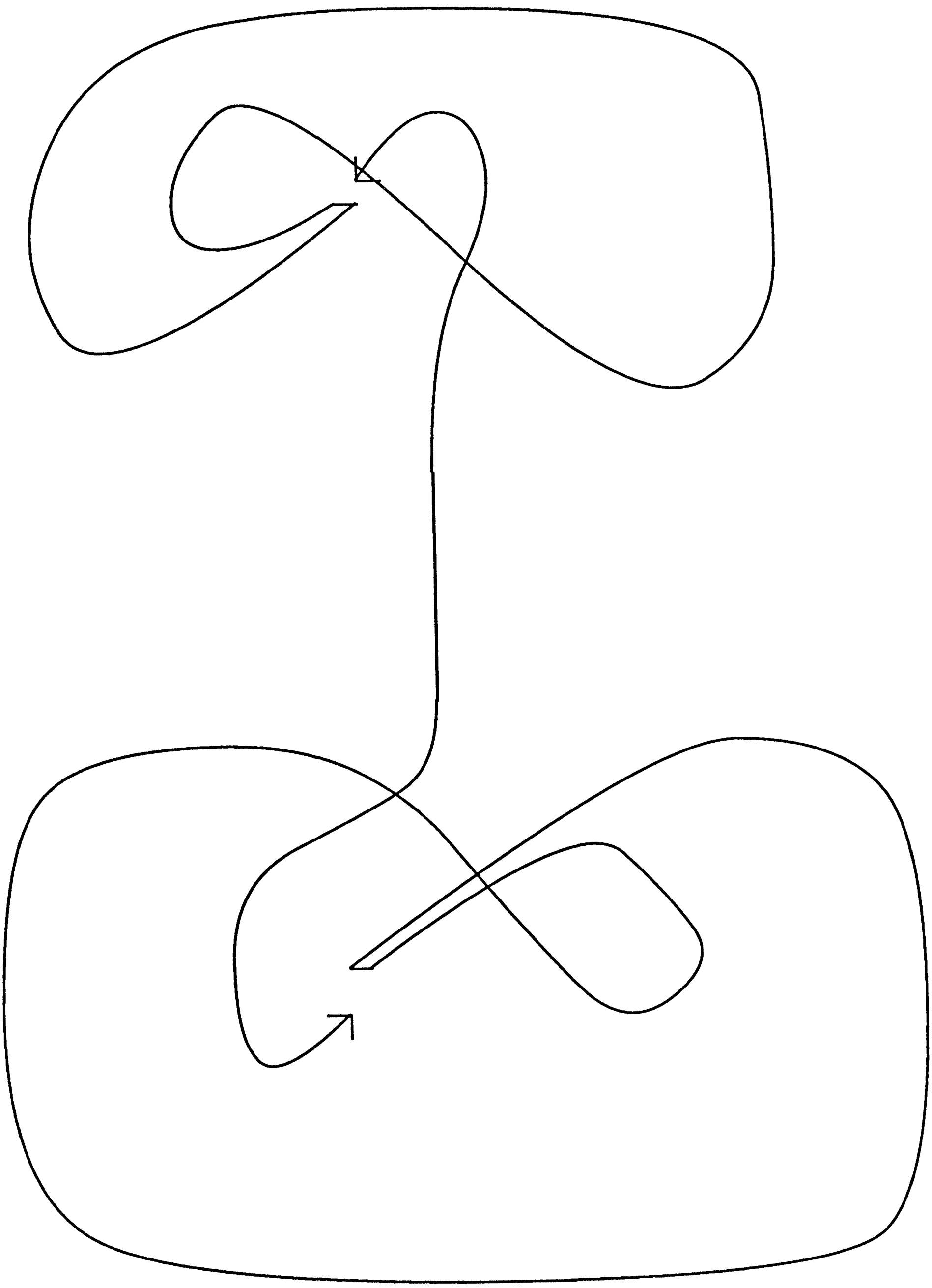


FIGURE 3: The Disk D_1

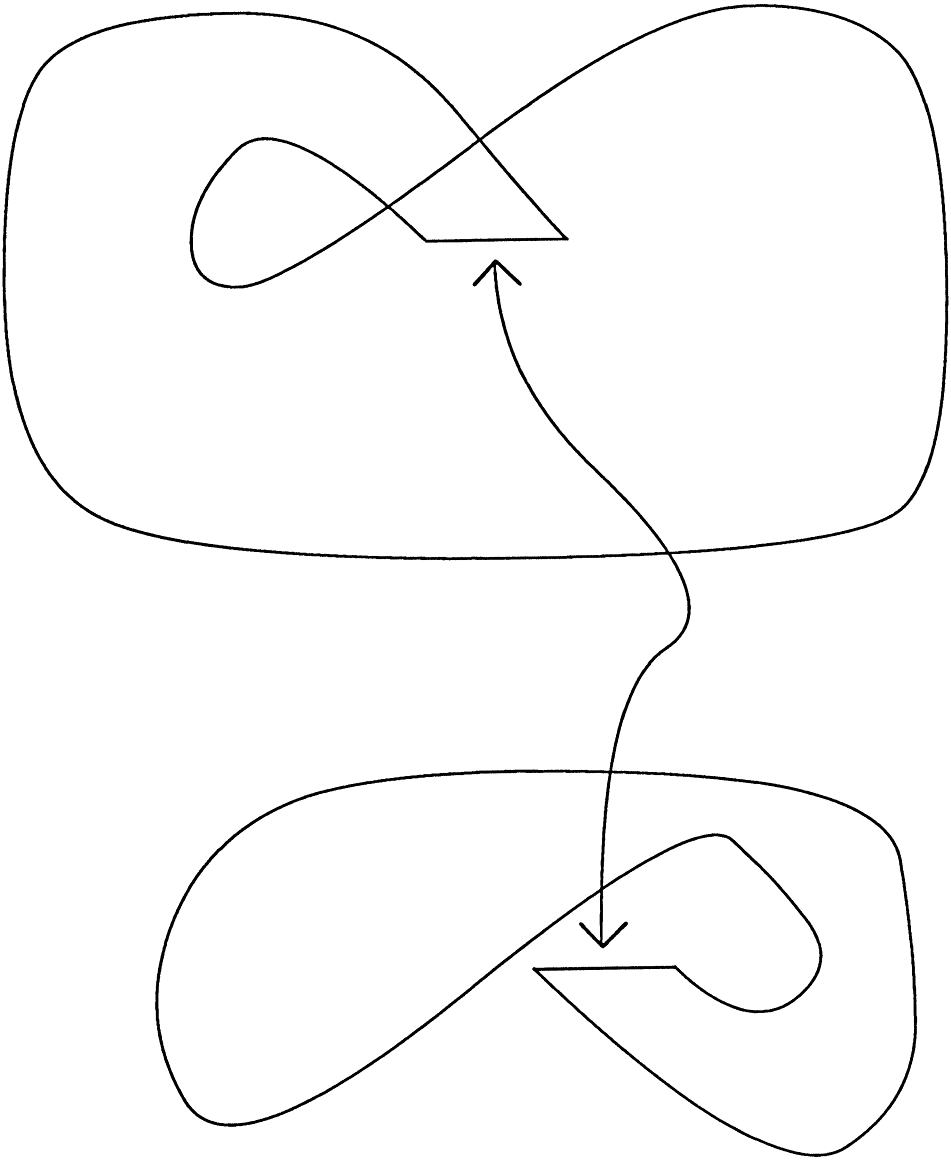


FIGURE 4: The Disk D_2

It is not difficult to see the two disks D_1 and D_2 bounded by this curve. We illustrate this in Figures 3 and 4 below, where, for ease in visualization, we have cut the disk D_1 (resp. D_2) into two pieces to reduce the overlapping.

Notice that D_1 and D_2 are isometric near their boundaries–indeed, a neighborhood of ∂D_1 in D_1 (resp. ∂D_2 in D_2) is given by a one-sided neighborhood of γ.

3. Neumann Boundary Conditions. In this section, we show that, for appropriate choices of γ, D_1 and D_2 are not isospectral for Neumann boundary conditions.

Our main tool is:

CHEEGER'S INEQUALITY (NEUMANN CASE) [4]. *For D a surface with boundary,*

$$\lambda_1^N(D) \geqslant \frac{1}{4}h^2,$$

where

$$h = \inf_L \frac{\text{length}(L)}{\min(\text{area}(A), \text{area}(B))},$$

as L runs over all curves dividing D into two pieces A and B.

Cheeger's inequality is, of course, valid for all dimensions, but for simplicity in notation we will restrict to the two-dimensional case.

We now choose γ so that D_1 has a long, narrow tube at the place which is cut open in Figure 3, but D_2 has no such narrow tube.

To estimate $\lambda_1^N(D_1)$, we consider a test function f_γ which is $\equiv c_1$ on one lobe in Figure 3, $\equiv -c_2$ on the other lobe, and which changes linearly along the narrow tube, where c_1 and c_2 are two positive constants chosen so that $\int_{D_1} f_\gamma = 0$. One choice is to let c_2 be approximately the area of the first lobe, and c_1 be approximately the area of the second lobe.

We now compute the Rayleigh quotient

$$\frac{\int \|\text{grad}(f_\gamma)\|^2}{\int \|f_\gamma\|^2}.$$

But $\|\text{grad}(f_\gamma)\|$ is supported only on the tube, and its value depends only on the length of the tube, not its width. It follows that as the width of the tube tends to 0, the Rayleigh quotient, and hence $\lambda_1^N(D_1)$, tends to 0 as well.

On D_2, however, there is no such narrow tube, and in the limit, D_2 converges to a smooth disk. It follows that $h(D_2)$ is bounded away from zero, and hence, by Cheeger's inequality, the same is true for $\lambda_1^N(D_1)$. It follows that, as the tube narrows, $\lambda_1^N(D_1)$ is strictly less than $\lambda_1^N(D_2)$.

4. Dirichlet Boundary Conditions. In this section, we will show how to choose γ so that D_1 and D_2 are not isospectral for Dirichlet boundary conditions.

We first remark that Cheeger's inequality is also available in this case:

CHEEGER'S INEQUALITY (DIRICHLET CASE) [4].

$$\lambda_0^D(D) \geqslant \frac{1}{4}h^2,$$

where

$$h = \inf_L \frac{\text{length}(L)}{\text{area}(\text{int}(L))},$$

as L runs over closed curves in D dividing D into two components, one of which does not contain ∂D. We call this component int(L).

It is possible to analyze the curve L which realizes this minimum—see [2] for a discussion—and from this arrange γ so that $\lambda_0^D(D_1) > \lambda_0^D(D_2)$. The following variant of Cheeger's inequality, due to Osserman [10], is much easier to work with:

THEOREM (OSSERMANN) [10]. *If D is a compact flat disk, then*

$$\lambda_0^D(D) \geqslant \frac{1}{4\rho^2},$$

where ρ is the radius of the largest disk contained in D.

Note that Osserman's theorem is usually stated for planar disks, but was proved in [10] in the more general case considered here, and indeed under the more general assumption that D is simply connected and has non-positive curvature.

Note also the upper bound

$$\lambda_0^D(D) \leqslant \frac{(\text{const})}{\rho^2},$$

where (const) is the lowest Dirichlt eigenvalue of the unit disk, as can be seen from the fact that λ_0^D is strictly decreasing under inclusion of domain.

We may now choose our curve γ in the following way, as shown in Figure 5, which shows the right-hand lobe of γ: the outside curve becomes a large lobe, while the inner curves wiggle in and out.

In Figures 6 and 7, we show the corresponding parts of the disks D_1 and D_2. Note that D_2 has a very large right-hand lobe, and hence $\lambda_0^D(D_2)$ is very small. However, the corresponding parts of D_1 contain no balls of large radius (they are destroyed by the wiggling). It follows from Osserman's theorem that $\lambda_0^D(D_1)$ is large, and in particular $\lambda_0^D(D_1) > \lambda_0^D(D_2)$.

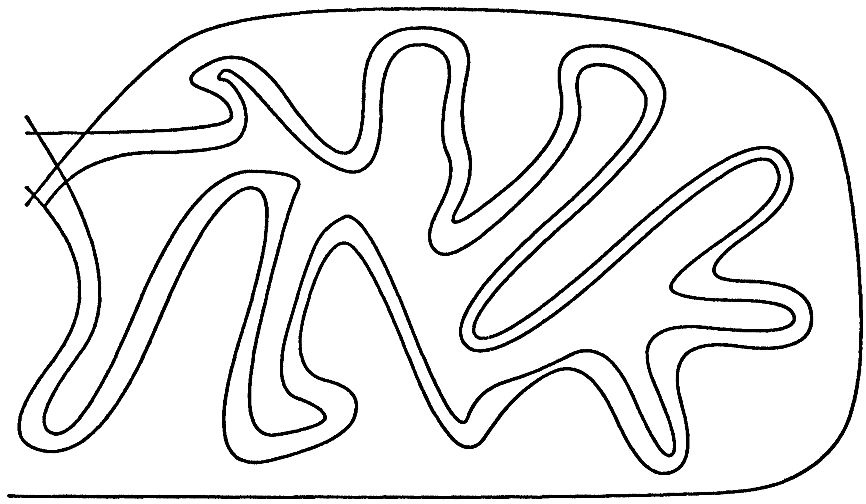


FIGURE 5

Note that this wiggling can be done independent of the narrowing of the tube of §3. In particular, this completes the proof of Theorem 1.

Another, less elementary, approach is given by the recent theorem of Osgood, Phillips, and Sarnak [9]. Their theorem is stated for plane domains, but at the first step they pass to flat matrics on the unit disk via the Uniformization Theorem, so their proof is valid without change to the category of compact flat disks. They show that as the boundary of a domain pinches in on itself so that the heat invariants remain bounded, then $-\log(\det(\Delta))$ must tend to $+\infty$, where $\det(\Delta)$ is the determinant of the Laplacian for Dirichlet boundary conditions, and is a spectral invariant.

From this we conclude that, in the construction we gave for Neumann boundary conditions in §3, as the tube narrows on D_1, D_1 and D_2 must differ spectrally for the Dirichlet as well as Neumann conditions, since $-\log(\det(\Delta))$ remains bounded for D_2. In this approach, though, one is hard put to decide which eigenvalues for the Dirichlet conditions differ.

We now turn to Theorem 2. We observe that the planar disks shown in Figure 8 below are evidently of the same area and piecewise isometric near the boundary. However, the disk on the right has a thin neck in the middle, while the one on the left does not, so they differ for Neumann boundary conditions. Furthermore, the left disk has a large embedded ball while the right one does not, so they differ also for Dirichlet conditions.

5. Higher Dimensions. In this section, we will show how the construction of §2 generalizes to give flat n-dimensional disks D_1^n and D_2^n which are isometric near their boundaries but are not isospectral.

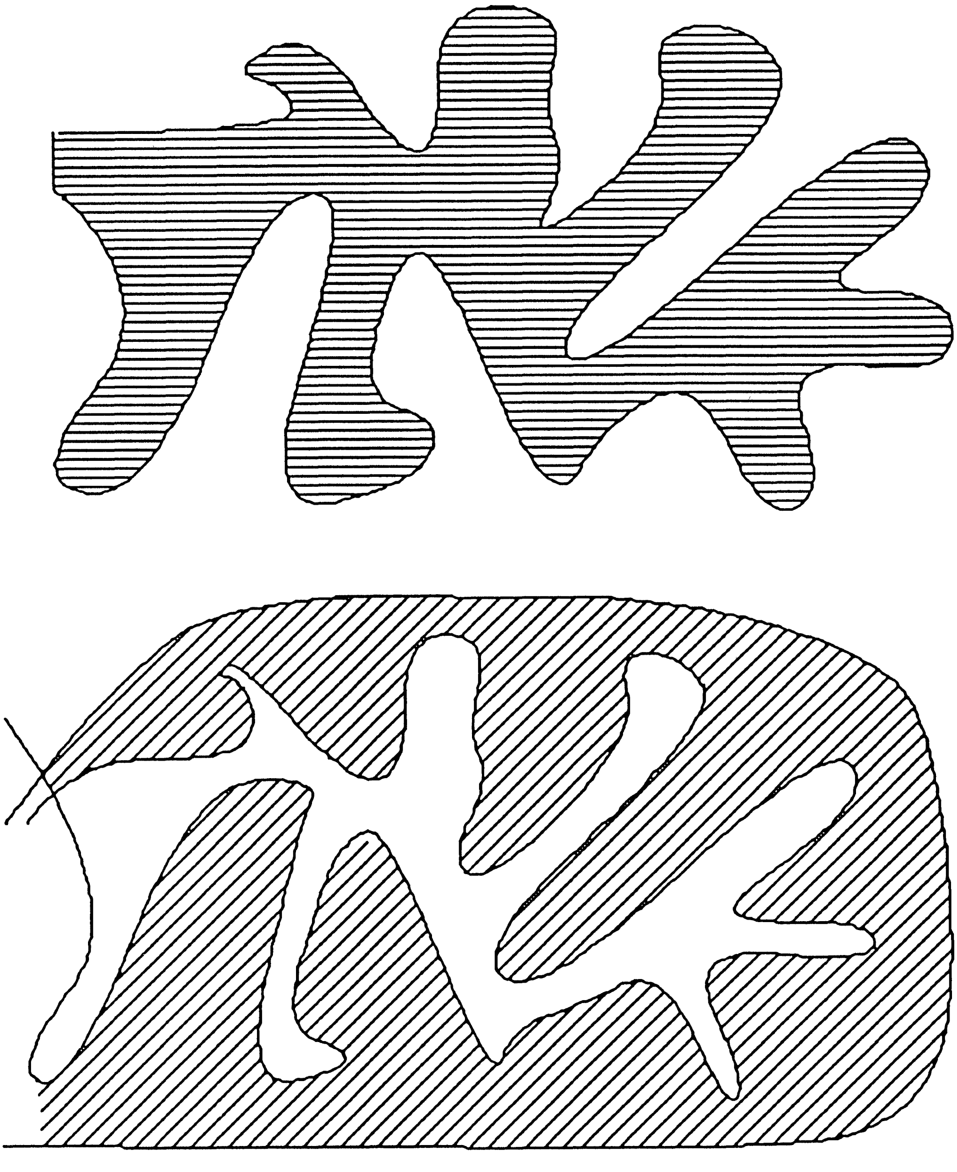
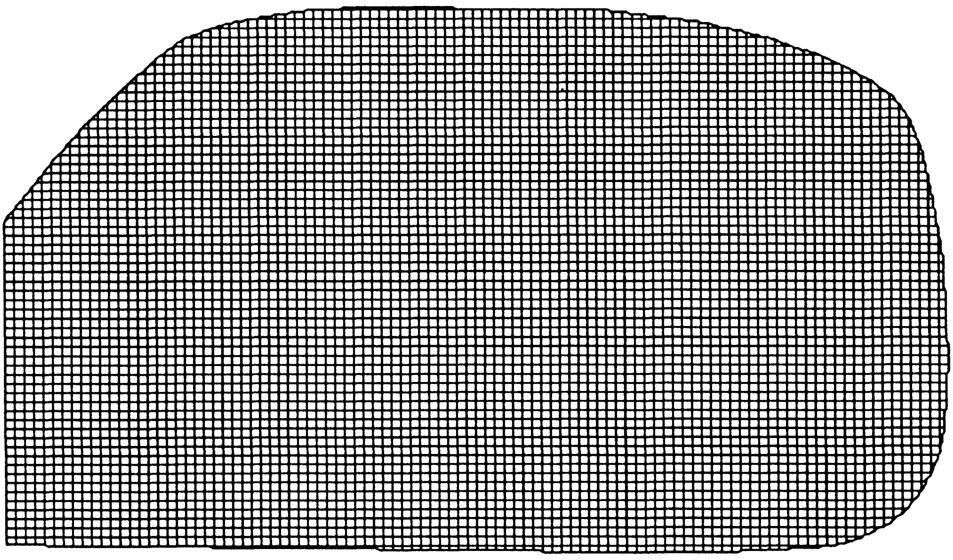


FIGURE 6: The wiggled D_1



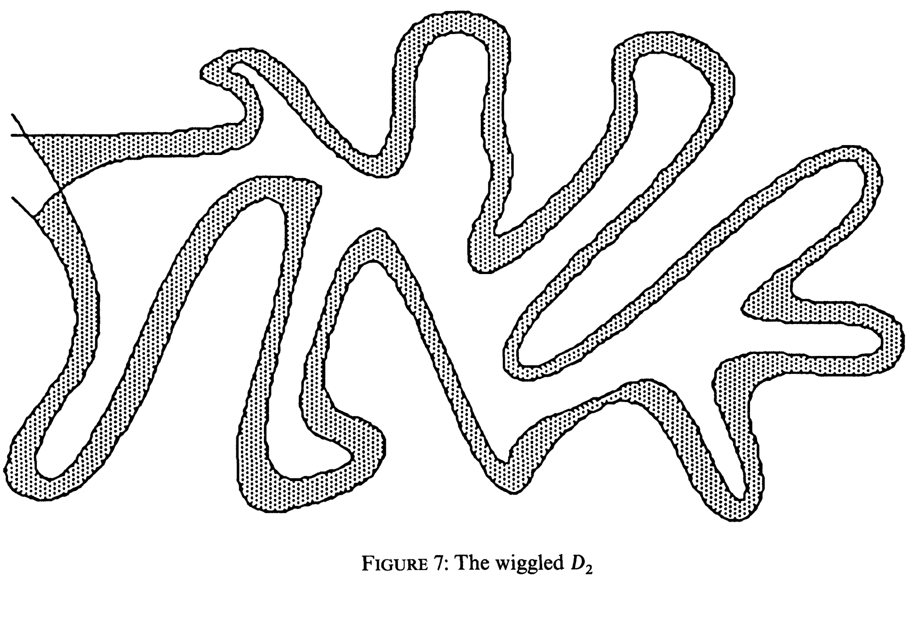


FIGURE 7: The wiggled D_2

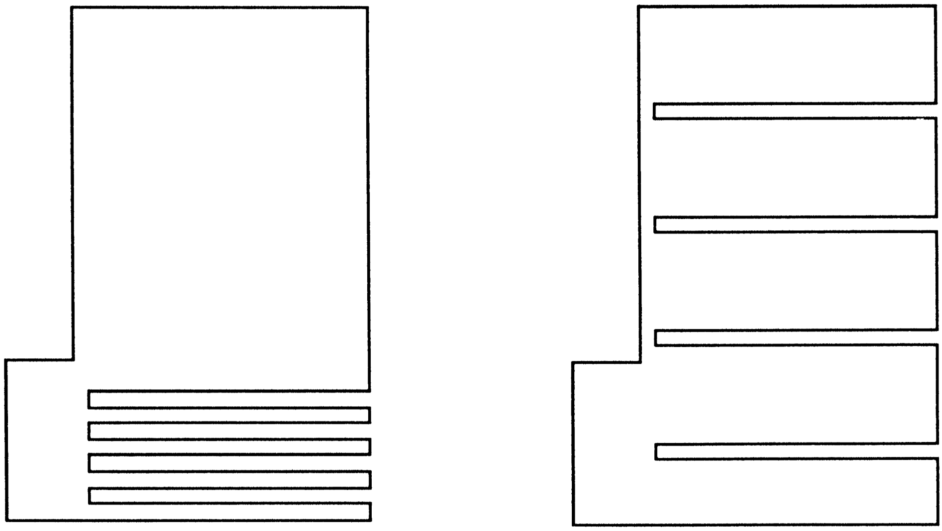


FIGURE 8

We begin with the following observation: let U be a region in $\mathbb{R}^2$ which is symmetric about the x_1-axis. Let us denote by U^n the region in $\mathbb{R}^n$ obtained from U by including $\mathbb{R}^2$ in $\mathbb{R}^n$, and rotating U by the orthogonal group $SO(n-1)$ keeping the x_1-axis fixed. It is then obvious that if U is topologically a disk, then U^n is topologically an n-dimensional disk, and its boundary is an $n-1$-dimensional sphere.

To construct D_1^n and D_2^n, we now proceed as follows: letting D be the standard disk in $\mathbb{R}^2$, we let U_1 be obtained by joining D with two copies of the disk D_1 of Theorem 1 placed symmetrically about the x_1-axis, and let U_2 be obtained analogously from D_2. Then the corresponding n-disks U_1^n and U_2^n are the desired disks D_1^n and D_2^n. Their common boundary is the $(n-1)$-sphere S which is the $SO(n-1)$-orbit of $\partial U_1 = \partial U_2$. Note that the intersection of D_1^n with any plane containing the x_1-axis is isometric to U_1, and the intersection of D_2^n with any plane containing the x_1-axis is isometric to U_2.

We now claim that D_1^n and D_2^n are spectrally distinct.

This is clear when we look at Neumann boundary conditions, for the same reasons as in §3. D_1^n can be cut into two pieces along the thin neck by a hypersurface of the form $\mathbb{D}^1 \times S^{n-2}$ and has small area, but D_2^n cannot be so divided.

Dirichlet conditions look more problematic, since both Osserman's theorem and the theorem of Osgood-Phillips-Sarnak are strictly two-dimensional arguments.

However, we recall the standard fact that λ_0^D is always a simple eigenvalue. It follows that the corresponding eigenfunction ϕ_0 is invariant under the natural $SO(n-1)$-action, since the same is true of D_1^n and D_2^n by construction. It follows that, for any point x, grad(ϕ_0) always points in direction contained in the plane containing x and the x_1-axis. Therefore, the Rayleigh quotient of ϕ_0 is given by the Rayleight quotient of ϕ_0 restricted to any plane containing the x_1-axis. This is then estimated by Osserman's theorem, exactly as in §4.

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Brooks: Department of Mathematics, University of Southern California, Los Angeles, California 90089-1113.

Eliashberg: Department of Mathematics, Stanford University, Stanford, California 94305-2125.

McMullen: Department of Mathematics, Princeton University, Princeton, New Jersey 08544.